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## Quantized calculus and Teichmüller space

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### 1. quantized calculus.

The famous duality theorem by Fefferman states that the dual space of  $\text{Re}H^1(S^1)$  is  $BMO(S^1)$ . On the other hand,  $H^1$  can be represented as a product of two elements in  $H^2(S^1)$ ,

$$h \in \text{Re}H^1 \longleftrightarrow h = g_1 H g_2 + (H g_1) g_2, \quad g_j \in L^2(S^1)$$

Here,  $H$  is the Hilbert transformation. Further Fefferman showed that

$$\left| \int f h d\theta \right| \leq C \|f\|_{BMO} \|g_1\|_2 \|g_2\|_2$$

Hence for every function  $f$  on  $S^1$ , set

$$[H, f](g) = H(fg) - fH(g) \quad g \in L^2(S^1),$$

and we have

$$\int f h d\theta = \int [H, f](g_1) g_2 d\theta.$$

**THEOREM ([1]).** *The operator  $[H, f]$  is bounded if and only if  $f \in BMO(S^1)$ .*

Following A. Connes, this operator is called the quantized derivative  $d^Q(f)$  of  $f$ . In fact, considering on the real line, we have

$$[H, f](g) = \text{Const.} \int_{\mathbb{R}} \frac{f(x) - f(y)}{x - y} g(y) dy$$

and hence can be considered as the "polarization" of usual differentiation.

Moreover we know

THEOREM ([1]). *The operator  $[H, f]$  is compact if and only if  $f \in VMO(S^1)$ .*

Here  $VMO(S^1)$  is the closure of  $C(S^1)$  in  $BMO(S^1)$ . In particular, if  $f \in C(S^1)$ , then  $[H, f]$  is compact. Recall that the dual space of  $VMOA(S^1)$  is  $H^1(S^1)$  and that an element of  $VMO(S^1)$  is not necessarily continuous but only 'quasicontinuous'. More precisely,

$$L^\infty \cap VMO = QC \left( = (H^\infty + C) \cap (\overline{H^\infty} + C) = C + HC \right)$$

REMARK ([2]). *If  $f \in L^\infty$  and  $|f| \in C(S^1)$ , then  $f \in QC$ .*

Now, "smoother"  $f(S^1)$  in fractal sense, better  $f$  as a compact operator.

THEOREM ([3]). *The operator  $[H, f]$  belongs to the Schatten class  $\mathfrak{L}^p$  if and only if  $f \in B_p^{1/p}(S^1)$ , where  $B_p^{1/p}(S^1)$  is the Besov space as below.*

Here  $f \in \mathfrak{L}^p$  means that the sequence of eigenvalues of  $|T| = (T^*T)^{1/2}$  belongs to  $\ell^p$ . (In particular,  $\mathfrak{L}^2$  is the Hilbert-Schmidt class.)

Next  $f \in B_p^{1/p}(S^1)$  means that  $f$  satisfies the inequality

$$\iint_{S^1 \times S^1} |f(x+t) - 2f(x) + f(x-t)|^p t^{-2} dx dt < +\infty.$$

Recall that, if  $p > 1$ , this inequality is equivalent to

$$\iint_{S^1 \times S^1} |f(x+t) - f(x)|^p t^{-2} dx dt < +\infty.$$

On the other hand, considering the harmonic extension on  $D$ ,  $f \in B_p^{1/p}(S^1)$  if and only if

$$\int_D \|D^2 f\|^p (1 - |z|)^{2p-2} |dz \wedge d\bar{z}| < +\infty.$$

(When  $p > 1$ , this is equivalent that  $f$  is  $p$ -integrable 1-form, namely

$$\int_D \|Df\|^p (1 - |z|)^{p-2} |dz \wedge d\bar{z}| < +\infty.)$$

COROLLARY.  $B_2^{1/2}(S^1)$  is the Sobolev space (the harmonic Dirichlet space)  $HD(D) = W_1^2(D) \cap H(D)$ , where  $D$  is the unit disk.

Boundary values form  $H^{1/2} = \{(a_n) \mid \sum |n| |a_n|^2 < +\infty\}$  (, which S. Nag used).

## 2. On Hausdorff dimension of quasicircles.

A Riemann map  $f$  onto a  $K$ -quasi disk has a  $(1/K)$ -Hölder continuous boundary value. Hence, for instance (, also see Astala, to appear), we have

PROPOSITION (CF. FALCONER). *The Hausdorff dimension of a  $K$ -quasicircle is at most  $2 - \frac{1}{K}$ .*

On the other hand,

THEOREM (SULLIVAN). *Assume that there is a cocompact quasiFuchsian group  $\Gamma$  whose limit set is a quasicircle  $C$  as the limit set. Then The Hausdorff dimension of  $C$  is  $p$  if and only if a Riemman map  $f$  onto the interior of  $C$  belongs to  $B_q^{1/q}(S^1)$  for every  $q > p$ .*

COROLLARY ([4]). *A quasicircle  $C$  as in Theorem 4 has Hausdorff dimension  $p$  if and only if*

$$p = \inf\{q \mid [H, f] \in \mathcal{L}^q\}.$$

PROBLEM. *Characterize such quasicircles that corresponds to finitely generated Kleinian groups.*

## 3. Teichmüller spaces.

Here we will give new representation of the Universal Teichmüller sapce. First we recall (cf. Astala-Gehring, '86) the follllowing

THEOREM (KOEBE).  $\{\log f' \mid f \text{ is univalent on } D\}$  is bounded in the Bloch space

$$\mathfrak{B} = \{f \mid \sup(1 - |z|^2)|f'(z)| < +\infty\}.$$

On the other hand, the boundary value of  $\log f'$ , where  $f \in T(1)$ , does not necessarily belong to  $BMO(S^1)$ . (cf. Astata-Zinsmeister, '91)

Now, if  $f$  is a Riemann map onto a quasidisk,  $f$  itself has a continuous boundary value. Hence we can consider to represent Riemann maps in the above spaces.

First we set

$$\Sigma = \{f \mid f \text{ is univalent on } D \text{ and has a form } = \frac{1}{z} + \sum_{n=1}^{\infty} c_n z^n \text{ near } z = \infty, \}$$

and equip  $\Sigma$  with the Bers topology. Then  $\Sigma$  has the subset  $\Sigma_1$  which we can identify with the universal Teichmüller space  $T(1)$ .

THEOREM.  $\Sigma$  can be mapped injectively in  $VMO(S^1)$ .

This injection is continuous at least on  $\Sigma_1$ .

In general,  $BMO(S^1) \subset \mathfrak{B}$  and hence  $VMO(S^1) \subset \mathfrak{B}_0$ , and it is known that, for  $g \in \mathfrak{B}_0$ ,  $g$  has a finite angular limit on a set of Hausdorff dimension 1 (Makarov '89). Also recall that  $AD(D) \subset VMOA(D)$  (S.Yamashita '82. Further, see Aulaskari '88), and that  $f \in \Sigma$  has a finite angular limit almost everywhere as is seen by classical Plessner's theorem.

On the other hand, Pommerenke ([6]) showed that, under the locally uniformly boundedness assumption of average multiplicity,

$$f \in BMOA(S^1) \text{ if and only if } f \in \mathfrak{B}, \quad f \in VMOA(S^1) \text{ if and only if } f \in \mathfrak{B}_0.$$

On the other hand, since multiplication by  $z$  is an invertible  $VMO$ -multiplier, we can identify  $\Sigma$  with  $z\Sigma \subset VMOA(S^1)$ . In particular, we have the following

COROLLARY.  $\Sigma$  can be mapped injectively in  $\mathfrak{B}_0$ .

*This injection is continuous at least on  $\Sigma_1$ .*

REMARK. Recall that a Riemann map has a continuous boundary value if and only if the complement is locally connected. Hence the locally connectedness conjecture of the limit set (cf. Abikoff([7])) can be restated as follows;

*The image of  $\Sigma(G)$  is contained in  $C(S^1)$  for a finitely generated Kleinian group  $G$ , where  $\Sigma(G)$  corresponds to  $T(G)$  ?*

*It seems interesting to characterize Riemann maps, or elements of  $\Sigma$ , belonging to  $VMO(S^1) - C(S^1)$  geometrically.*

Now to prove Theorem, we note the following fact, which follows at once from the equivalence of  $VMO(S^1)$  and  $\mathfrak{B}_0$ , and from the geometrical characterization of Bloch functions by Pommerenke ([5]).

PROPOSITION. *Let  $f$  be a holomorphic injection of  $D$ . If  $f(D)$  is bounded, then the boundary value  $f$  belongs to  $VMO(S^1)$ .*

But this fact has an interesting

COROLLARY. *Let  $G$  be any Kleinian group which has  $\infty$  as an ordinary point, and  $f$  be a Riemann map onto a simply connected component of  $G$ . Then  $f \in VMO(S^1)$ .*

Here we note that  $VMO$ -ness is a local property.

LEMMA (GOTOH). *Let  $f$  be meromorphic on  $D$  and has no poles near  $\partial D$ . If, for every  $\zeta \in \partial D$ , there is a neighborhood  $U$  of  $\zeta$  such that  $f \circ \phi_\zeta \in VMOA(S^1)$ , where  $\phi_\zeta$  is a Riemann map onto  $U \cap D$ , then  $f \in VMOA(S^1)$ .*

COROLLARY. *Let  $f$  be a meromorphic injection of  $D$ . If  $\infty \in f(D)$ , then the boundary value  $f$  belongs to  $VMO(S^1)$ .*

PROOF OF THEOREM: Since the injectivity is clear, the first assertion follows from the above Corollary.

Next suppose that  $f_n$  converges to  $f$  in  $\Sigma_1$ . Then by uniform convergence property of normalized quasiconformal maps, we can see that  $f_n$  converges to  $f$  uniformly on  $\overline{D}$ . In particular,  $f_n$  converges to  $f$  in  $L^\infty(S^1)$  and hence in  $BMO(S^1)$ , which shows the second assertion, continuity of injection on  $\Sigma_1$ .

REMARK. *Local character of functions in  $VMO(S^1)$  can be restated as Axler-Shapiro's theorem ([8]). On the other hand, when  $C$  is the limit set of a b-group, every prime end of the invariant component is area 0 by Ahlfors-Thurston's 0 – 1 theorem. Hence these facts give another proof of the above Theorem for this case.*

PROBLEM. *Is the above injection, say  $E$ , continuous on the whole  $\Sigma$  ? If not, determine the corona, i.e. the set  $\overline{E(\Sigma_1)} - E(\overline{\Sigma_1})$ .*

Some further discussion on this problem will appear elsewhere.

Next, another representation can be obtained by considering the set

$$\tilde{S} = \{f \mid f \text{ is univalent and holomorphic on } D\}$$

Again we write as  $\tilde{S}_1$  the set corresponding to  $T(1)$ , namely, the set of Riemann maps which admits a quasiconformal extension. Then the 'VMO-ness at a point' can measure the local complexity at the point metrically. For instance, we have

PROPOSITION. *Suppose that  $f$  is a Riemann map onto a component  $B$  of a Kleinian group  $G$ . If  $\infty$  belongs to the boundary of  $B$  and is fixed by an element of  $G$  with infinite order, then  $f$  does not belongs to  $BMO(S^1)$ .*

PROOF: If  $\infty$  is a parabolic fixed point, then the existence of a cusp neighborhood implies that  $f \notin BMO(S^1)$  by Pommerenke's characterization of Bloch functions

([5]).

If  $\infty$  is a loxodromic fixed point, then from self-similarity (invariance) of the limit set, we can conclude the assertion again by Pommerenke's characterization.

Outside of the fixed points set, the limit set of  $G$  may have high complexity, at least, in the finitely generated case. Hence the Riemann map  $f$  may also behave very wildly. So we may put the following

**PROBLEM.** *If  $G$  is a finitely generated Kleinian group with a component  $f(D)$  and  $\infty$  is not fixed by any non-trivial element of  $G$ , does  $f$  belong to  $VMO(S^1)$  ?*

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